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Class Groups of n-Noetherian Rings[1]

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INTRODUCTION

Let A be a noetherian integrally closed integral domain. Let $\mathscr{N}$ denote the category of finitely generated A-modules and let $\mathscr{N}'$ denote the Serre subcategory of $\mathscr{N}$ consisting of all M such that $M_{\mathfrak{p}} = 0$ for all height one prime ideals $\mathfrak{p}$ of A. As described in ([*1*], Chap. VII), the Grothendieck group of the quotient category $\mathscr{N}/\mathscr{N}'$—$K^0(\mathscr{N}/\mathscr{N}')$—is isomorphic to the direct sum of a free cyclic group and the ideal class group of A. Many factorization problems are best attacked by means of the isomorphism $K^0(\mathscr{N}/\mathscr{N}') \simeq \mathbf{Z} \oplus C(A)$, where $C(A)$ denotes the ideal class group of A.

Although $C(A)$ is defined for any Krull domain A, there has apparently not been a comparable theorem for the case of a (non-noetherian) Krull domain. We attempt to remedy this in the present paper. If A is a Krull domain, $\mathscr{M}$ denotes the category of all unitary A-modules, and $\mathscr{M}'$ the Serre subcategory of $\mathscr{M}$ consisting of all M, such that $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of A of height one, then we may form the quotient category $\mathscr{M}/\mathscr{M}'$. Let $(\mathscr{M}/\mathscr{M}')^{\#}$ denote the (exact abelian) full subcategory of $\mathscr{M}/\mathscr{M}'$ consisting of all noetherian objects of $\mathscr{M}/\mathscr{M}'$. Then we have that $K^0((\mathscr{M}/\mathscr{M}')^{\#}) = \mathbf{Z} \oplus C(A)$. The essential point is that A (as an object of $\mathscr{M}/\mathscr{M}'$) is a noetherian object of $\mathscr{M}/\mathscr{M}'$.

In an earlier article [*2*] we have defined groups $W_i(A)$ $(i \geqslant 1)$, which generalize to prime ideals of height i the class group of A, (assuming A is a noetherian ring). Since these groups are defined by means of Grothendieck groups of certain categories, it is possible, as in the case of a Krull domain above, to relax the requirement that A be a noetherian ring, to the requirement that A be a noetherian object in a certain quotient category. This leads us to the concept of an n-noetherian ring.

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This in turn is a special case of the following situation: Let A be a ring (commutative), and $\mathscr{P}$ a collection of prime ideals of A. Let $\mathscr{A}$ denote the category of A-modules, and let $\mathscr{C}$ denote the Serre subcategory consisting of all M in $\mathscr{A}$, such that $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathscr{P}$. We say that A is noetherian with respect to the set $\mathscr{P}$ if A is a noetherian object in the quotient category $\mathscr{A}/\mathscr{C}$.

To give an adequate treatment of the groups $W_i(A)$, it has turned out to be convenient to be fairly systematic about the properties of $\mathscr{P}$-noetherian rings. Although many of the results are easy to anticipate, the details are occasionally tricky and we have given complete proofs in all cases in which the results do not follow in a straightforward fashion.

Section 1 recalls the necessary background pertaining to quotient categories and adjoint functors to be found in [3]. Section 2 gives some criteria for a ring A to be noetherian with respect to a set $\mathscr{P}$ of prime ideals of A, and Section 3 gives some inheritance properties important in the theory of class groups. In Section 4 we develop a primary decomposition theory adequate for our current needs. Section 5 is devoted to Krull domains; in particular several results concerning lattices found in [1] are generalized to Krull domains. In Section 6 we simply state some special cases of results of Section 3 in the case where A is $\mathscr{P}_n$-noetherian ($\mathscr{P}_n$ being the set of prime ideals of A of height at most n). Section 7 is devoted to the original problem of the definition and properties of the groups $W_i(A)$ $(1 \leqslant i \leqslant n)$ for a ring A which is $\mathscr{P}_n$-noetherian. In particular, the isomorphism $K^0((\mathscr{M}/\mathscr{M}')^{\#}) = \mathbf{Z} \oplus C(A)$ for a Krull domain A is obtained. A further feature is the definition of an n-flat extension B of A for which homomorphisms $W_i(A) \to W_i(B)$ $(1 \leqslant i \leqslant n)$ are naturally defined which generalizes the condition given in [1] for such maps to be defined for Krull domains A and B.

1. Introductory Material

We begin by recalling the main facts concerning quotient categories of an abelian category; see ([3], Chap. III) for proofs and a systematic development.

Let $\mathscr{A}$ be a concrete abelian category, and let $\mathscr{C}$ be a *Serre* subcategory of $\mathscr{A}$, i.e. a full subcategory of $\mathscr{A}$ such that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in $\mathscr{A}$, then M_1 and M_3 are in $\mathscr{C}$ if, and only if, M_2 is in $\mathscr{C}$. One may form a new abelian category $\mathscr{A}/\mathscr{C}$ (the *quotient category* of $\mathscr{A}$ by the Serre subcategory $\mathscr{C}$) whose objects are the objects of $\mathscr{A}$ and with

$$\mathrm{Hom}_{\mathscr{A}/\mathscr{C}}(M, N) = \varinjlim \mathrm{Hom}_{\mathscr{A}}(M', N/N')$$

where the limit is taken over the set (M', N') with M/M' and N' in $\mathscr{C}$. The functor $T: \mathscr{A} \to \mathscr{A}/\mathscr{C}$ defined by $T(M) = M$ on objects and $T(f) =$ class of f in $\mathrm{Hom}_{\mathscr{A}/\mathscr{C}}(M, N)$ for f in $\mathrm{Hom}_{\mathscr{A}}(M, N)$ is an exact functor from $\mathscr{A}$ to $\mathscr{A}/\mathscr{C}$. Furthermore, if U is any exact functor from $\mathscr{A}$ to an abelian category $\mathscr{B}$ for which $U(M) = 0$ for all M in $\mathscr{C}$, then there is a (unique up to isomorphism) functor $V: \mathscr{A}/\mathscr{C} \to \mathscr{B}$ such that $U = VT$.

The exactness of T, of course, means that T will preserve (finite) biproducts, intersections, and "unions". We recall also [3] that if $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence in $\mathscr{A}/\mathscr{C}$, and M_2 is an object of $\mathscr{A}$ such that $T(M_2)$ is isomorphic to N_2 (in $\mathscr{A}/\mathscr{C}$), then it is possible to find objects M_1, M_3 in $\mathscr{A}$ and isomorphisms α_1, α_2, α_3 in $\mathscr{A}/\mathscr{C}$, such that $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact in $\mathscr{A}$ and

$$\begin{tikzcd}
0 \arrow[r] & T(M_1) \arrow[r] \arrow[d, "\alpha_1"] & T(M_2) \arrow[r] \arrow[d, "\alpha_2"] & T(M_3) \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram in $\mathscr{A}/\mathscr{C}$.

An easy consequence of this is that if N is an object of $\mathscr{A}/\mathscr{C}$, and $N_0 \subseteq N_1 \subseteq \cdots$ is an ascending sequence of subobjects of N in $\mathscr{A}/\mathscr{C}$, then for any M in $\mathscr{A}$ such that $T(M)$ is isomorphic to N in $\mathscr{A}/\mathscr{C}$, we can find an ascending sequence of subobjects of M, $M_0 \subseteq M_1 \subseteq \cdots$ such that $T(M_i)$ is isomorphic to N_i in $\mathscr{A}/\mathscr{C}$ for all i, and the isomorphisms commute with the imbeddings.

When the functor T has a right adjoint $S: \mathscr{A}/\mathscr{C} \to \mathscr{A}$, then T preserves colimits; in particular T preserves arbitrary coproducts, arbitrary "unions" and direct limits.

For our applications of quotient categories, $\mathscr{A}$ will be the category of unitary modules over a commutative ring A (with identity). Since such a category has enough injectives, the functor T will have an adjoint S if, and only if, the following condition holds: *If M is any object in $\mathscr{A}$, then there is a maximal subobject M' of M lying in $\mathscr{C}$.*

Following Gabriel, we say that an object M of $\mathscr{A}$ is *$\mathscr{C}$-closed*, if 0 is the maximal subobject of M lying in $\mathscr{C}$, and every exact sequence

$$0 \to M \to N \to P \to 0$$

with P in $\mathscr{C}$ splits. To construct an adjoint, let M be an object of $\mathscr{A}/\mathscr{C}$. Regard M as an object in $\mathscr{A}$, let M' be the maximal subobject of M in $\mathscr{C}$, imbed $M'' = M/M'$ in a $\mathscr{C}$-closed object N in $\mathscr{A}$ (e.g. take N to be the injective envelope of M''), and set $S(M)$ equal to the maximal subobject of N such that $S(M)/M''$ is in $\mathscr{C}$ ([3], Chap. III, Section 2, Proposition 4, p. 342).

We now apply these considerations to the following situation. Let A be a commutative ring and $\mathscr{A}$ the category of A-modules. Let $\mathscr{P}$ be a set of prime ideals of the ring A. Let $\mathscr{C}$ be the full subcategory of $\mathscr{A}$ consisting of those A-modules M, such that $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathscr{P}$. We note that we may as well assume that $\mathscr{P}$ is saturated in the following sense: If $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals in A with $\mathfrak{q} \subseteq \mathfrak{p}$, and $\mathfrak{p}$ in $\mathscr{P}$, then $\mathfrak{q}$ is in $\mathscr{P}$. It is clear that if M is an object of $\mathscr{A}$, then

$$M' = \{m \in M : \forall \mathfrak{p} \in \mathscr{P}, \exists s \notin \mathfrak{p}, sm = 0\} = \bigcap_{\mathfrak{p} \in \mathscr{P}} \operatorname{Ker}(M \to M_{\mathfrak{p}})$$

is the maximal subobject of M lying in $\mathscr{C}$.

We now give a definite construction for an adjoint functor $S : \mathscr{A}/\mathscr{C} \to \mathscr{A}$. Let M be an object of $\mathscr{A}/\mathscr{C}$, and regard M as an object of $\mathscr{A}$. Consider the homomorphism $M \to \prod M_{\mathfrak{p}}$. The kernel of this homomorphism is clearly the maximal subobject of M in $\mathscr{C}$ and we now show that $\prod M_{\mathfrak{p}}$ is $\mathscr{C}$-closed. It is clear that the maximal subobject of $\prod M_{\mathfrak{p}}$ in $\mathscr{C}$ is 0. Suppose that $0 \to \prod M_{\mathfrak{p}} \xrightarrow{f} L \to P \to 0$ is exact in $\mathscr{A}$, where P is in $\mathscr{C}$. For each element x of L and prime ideal $\mathfrak{p}$ in $\mathscr{P}$, there is an $s_{\mathfrak{p}} \notin \mathfrak{p}$ such that $s_{\mathfrak{p}}x \in \prod M_{\mathfrak{p}}$, say $s_{\mathfrak{p}}x = (n_{\mathfrak{q}})$, each $n_{\mathfrak{q}} \in M_{\mathfrak{q}}$. Set $g(x) = (n_{\mathfrak{p}}/s_{\mathfrak{p}})$. One checks that this assignment is unique (hence an A-homomorphism) and that $gf = 1$. We now set $S(M) = \{y \in \prod M_{\mathfrak{p}} : \forall\, \mathfrak{p} \in \mathscr{P}, \exists\, s \notin \mathfrak{p}$ such that $sy \in M\}$. Then S is an adjoint to T ([3], Chap. III, Section 2).

If we set $\mathscr{F} = \{I : I$ is an ideal of A and A/I is in $\mathscr{C}\}$, then $\mathscr{F}$ is a topologizing and idempotent set of ideals of A in the sense of ([3], Chapitre V, Section 2), and $ST(M)$ is (up to isomorphism) $M_{\mathscr{F}}$. In particular, there is a correspondence between the prime ideals $\mathfrak{p}$ of A and $ST(\mathfrak{p})$ of $ST(A)$, such that if $\mathfrak{p}$ is in $\mathscr{P}$, $ST(\mathfrak{p})$ is a proper prime ideal of $ST(A)$.

At this point, we introduce a notation which is quite suggestive. If $i : N \to M$ is an imbedding in $\mathscr{A}$ such that $T(i)$ is an isomorphism we will write $TN = TM$.

2. $\mathscr{P}$-NOETHERIAN RINGS

Let $\mathscr{P}$ be a saturated set of prime ideals of the ring A. We say A is *noetherian relative* to $\mathscr{P}$, or *$\mathscr{P}$-noetherian*, if TA is a noetherian object in the category $\mathscr{A}/\mathscr{C}$.

THEOREM 2.1. *Let $\mathscr{P}'$ and $\mathscr{P}$ be two (saturated) sets of prime ideals of A with $\mathscr{P}' \subseteq \mathscr{P}$. If A is $\mathscr{P}$-noetherian, then A is $\mathscr{P}'$-noetherian.*

Proof. Let $\mathscr{C}'$ and $\mathscr{C}$ be the Serre subcategories of A associated with $\mathscr{P}'$ and $\mathscr{P}$ respectively. Clearly $\mathscr{C} \subseteq \mathscr{C}'$. Let T and T' be the functors from $\mathscr{A}$ to

$\mathscr{A}/\mathscr{C}$ and $\mathscr{A}/\mathscr{C}'$. Then there is a functor $U : \mathscr{A}/\mathscr{C} \to \mathscr{A}/\mathscr{C}'$ such that $UT = T'$. Let $\{J_i\}$ be an ascending sequence of subobjects of $T'A$. Choose an ascending sequence of ideals of A, $\{I_i\}$, such that $T'I_j = J_j$ for each j. Then the sequence $\{TI_j\}$ stabilizes at some k, and therefore so does the sequence $\{UTI_j\}$.

COROLLARY 2.2. *If A is $\mathscr{P}$-noetherian and $\mathfrak{p} \in \mathscr{P}$, then $A_\mathfrak{p}$ is a noetherian ring.*

Proof. Let $\mathscr{P}'$ denote the set of prime ideals of A which lie in $\mathfrak{p}$. Obviously the associated Serre subcategory $\mathscr{C}' = \{M \in A : M_\mathfrak{p} = 0\}$. Since $M_\mathfrak{p}$ is a $\mathscr{C}'$-closed object, we may set $S'M = M_\mathfrak{p}$. Thus S' is exact in this case. Therefore the subobjects of $T'A$ and the $A_\mathfrak{p}$-modules of $A_\mathfrak{p}$ are in one-to-one order preserving correspondence, so $T'A$ being noetherian implies that $S'A = A_\mathfrak{p}$ is noetherian.

Let $\mathscr{P}$ be a saturated set of prime ideals of A. Since the functor T has a right adjoint one checks that TA is a generator in the category $\mathscr{A}/\mathscr{C}$. Thus when TA is noetherian, an object M of $\mathscr{A}/\mathscr{C}$ is noetherian if, and only if, there is an exact sequence $(TA)^n \to M \to 0$. We say that an A-module M is *$\mathscr{P}$-finitely generated* if there is a finite set of elements $\{x_i\}$ in M such that $T(\sum Ax_i) = TM$.

THEOREM 2.3. *Let $\mathscr{P}$ be a (saturated) set of prime ideals of A. An A-module M is such that TM is noetherian if, and only if, every submodule of M is $\mathscr{P}$-finitely generated.*

Proof. Suppose every submodule of M is $\mathscr{P}$-finitely generated and let $\{N_i\}$ be an ascending sequence of subobjects of TM. Let $\{M_i\}$ be an ascending sequence of subobjects of M such that $TM_i = N_i$. Set $M' = \bigcup M_i$ and assume that there are elements $x_1, \ldots, x_n$ in M' such that $T(\sum Ax_i) = TM$. Then if $\{x_i\} \subseteq M_k$ we have $TM_k = TM'$. If $N_k = TM_k$, then $N_k = TM_k = TM' = T(\bigcup M_i) = \bigcup TM_i = \bigcup N_i$, so the initial ascending sequence terminates.

Conversely suppose that TM is noetherian. Choose N maximal in M with the property that N is $\mathscr{P}$-finitely generated by the usual Zorn's lemma argument using the fact that TM is noetherian. Then $N = M$, for if not there would be an element x in M not in N but $N + Ax$ is still $\mathscr{P}$-finitely generated, a contradiction.

THEOREM 2.4. *A is $\mathscr{P}$-noetherian if and only if every prime ideal $\mathfrak{p}$ of A is $\mathscr{P}$-finitely generated.*

Proof. One implication follows from Theorem 2.3.

Now suppose that every prime ideal of A is $\mathscr{P}$-finitely generated. If $\{I_\alpha\}$ is a chain of ideals of A such that each I_α is not $\mathscr{P}$-finitely generated, then clearly $\bigcup I_\alpha$ is not $\mathscr{P}$-finitely generated. Therefore we may choose I maximal with respect to the property of not being $\mathscr{P}$-finitely generated. Then I must be a prime ideal. If not, choose $b, c \in A$ such that $b, c \notin I$, but $bc \in I$. By the maximality of I, $I : Ab$ and $I + Ab$ are $\mathscr{P}$-finitely generated, so choose $x_1, \ldots, x_k$ in I such that $T(ab + \sum Ax_i) = T(I + Ab)$ and $y_1, \ldots, y_m$ in $I : Ab$ such that $T(\sum Ay_i) = T(I : Ab)$. Then (compare [4], Theorem 3.4, p. 8) $T(\sum Ax_i + \sum Aby_j) = TI$, a contradiction. Therefore I must be a prime ideal of A so the hypothesis insures that I is $\mathscr{P}$-finitely generated, again a contradiction, and the proof is complete.

We end this section with two examples.

Example. 2.4. Let A be a ring with only a finite number of minimal prime ideals, $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. Suppose that $A_{\mathfrak{p}_i}$ is a noetherian ring for each i. Then A is noetherian relative to the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ of prime ideals of A. For let $\mathfrak{p}_i$ be such a prime ideal and choose $x \in \mathfrak{p}_i$, $x \notin \mathfrak{p}_j$, and $j \neq i$. Let $\mathfrak{p}_i A_{\mathfrak{p}_i} = \sum A_{\mathfrak{p}_i} y_v$, where the y_v are taken in A. Then, clearly $T(Ax + \sum Ay_v) = T\mathfrak{p}_i$. If $\mathfrak{q}$ is not a minimal prime ideal, then there is an element w in $\mathfrak{q}$ but in no $\mathfrak{p}_j$. Suppose $\mathfrak{q} \supseteq \mathfrak{p}_i$. Then $T(aw + Ax + \sum Ay_v) = TA = T\mathfrak{q}$, so all prime ideals are $\mathscr{P}$-finitely generated, and hence the statement follows from the previous theorem.

Example. 2.5. Let A be an integral domain and let $\mathscr{P}$ denote the collection of prime ideals of A of height less than or equal to one. Then A is $\mathscr{P}$-noetherian if $A_\mathfrak{p}$ is a noetherian ring for each prime ideal of height one of A and each non-zero $x \in A$ is contained in only a finite number of prime ideals of height one of A. For let $\mathfrak{p} \neq (0)$ be in $\mathscr{P}$. Let $0 \neq x \in \mathfrak{p}$ and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the other prime ideals of A of height one containing Ax. Choose $y \in \mathfrak{p}$, $y \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_t$. Choose $z_1, \ldots, z_s$ in $\mathfrak{p}$ such that $\sum A_\mathfrak{p} x_j = \mathfrak{p} A_\mathfrak{p}$. Then again $T\mathfrak{p} = T(Ax + Ay + \sum Az_i)$. That the other prime ideals of A are $\mathscr{P}$-finitely generated follows as in Example 2.4.

Remark. One can see directly that the conditions given in the examples are necessary as well as sufficient for A to be $\mathscr{P}$-noetherian, but this will be a by-product of the considerations of Section 4.

3. Inheritance Properties

THEOREM 3.1. *Let A be $\mathscr{P}$-noetherian, and let S be a multiplicatively closed subset of A. Let $\mathscr{P}'$ be the set of prime ideals of A not meeting S. Let $\mathscr{Q}$ denote the set of prime ideals of A_S which are extensions of the prime ideals of $\mathscr{P}'' = \mathscr{P} \cap \mathscr{P}'$. Then A_S is $\mathscr{Q}$-noetherian.*

Proof. Let $\mathscr{C}$, $\mathscr{C}'$, and $\mathscr{C}''$ be the Serre subcategories of $\mathscr{A}$ determined by $\mathscr{P}$, $\mathscr{P}'$, and $\mathscr{P}''$ respectively, and let $T : \mathscr{A}\ \mathscr{A}/\mathscr{C}$, $T' : \mathscr{A}/\mathscr{C}'$, and $T'' : \mathscr{A}\ \mathscr{A}/\mathscr{C}''$ denote the functors to the respective quotient categories. Then there are functors $U' : \mathscr{A}/\mathscr{C}\ \mathscr{A}/\mathscr{C}''$ and $U'' : \mathscr{A}/\mathscr{C}''\ \mathscr{A}/\mathscr{C}''$ such that $U'T = T'' = U'T'$. Now $\mathscr{A}/\mathscr{C}'$ is isomorphic to the category $\mathscr{B}$ of A_S-modules (compare the proof of Corollary 2.2.), and U'' is equivalent to the functor derived from the set $\mathscr{Q}$ of prime ideals of A_S. Since $T''A$ is noetherian therefore $U''TA$ is noetherian, and the theorem is established.

THEOREM 3.2. *Let A be $\mathscr{P}$-noetherian, and let $\mathscr{P}'$ be the set of prime ideals $\mathfrak{P}$ of $A[X]$ such that $\mathfrak{P} \cap A \in \mathscr{P}$. Then $A[X]$ is $\mathscr{P}'$ noetherian.*

Proof. (after the first proof of the Hilbert basis theorem in Zariski-Samuel, vol. I). Let $\{I_j\}$ be an ascending sequence of ideals of $A[X]$. We will show that the sequence $\{T'I_j\}$ stabilizes.

If $\mathfrak{a}$ is an ideal of $A[X]$, let $\mathfrak{a}_n$ be the ideal (in A) of the coefficients of X^n in the polynomials of degree $\leqslant n$ in $\mathfrak{a}$. Suppose now that $\mathfrak{a} \subseteq \mathfrak{b}$, and $T\mathfrak{a}_n = T\mathfrak{b}_n$ for all $n \geqslant 0$. Then $T'\mathfrak{a} = T'\mathfrak{b}$. For let $g(X) = b_kX^k + \cdots + b_0$ be in $\mathfrak{b}$ with $b_k \neq 0$. If $\mathfrak{P} \in \mathscr{P}'$, then $\mathfrak{p} = \mathfrak{P} \cap A \in \mathscr{P}$, so there is s in $A - \mathfrak{p}$ with $sb_k \in a_k$. Let $f(X)$ be in $\mathfrak{a}$ such that $f(X) = sb_kX^k + \cdots + a_0$. Then $sg - f$ is in $\mathfrak{b}$ but of lower degree than k. Continuing, we get that there is some $t \in A - \mathfrak{p}$ such that $tg \in \mathfrak{a}$. Since $t \notin \mathfrak{P}$, this shows that $T'\mathfrak{b} = T'\mathfrak{a}$.

Now consider the double sequence $\{TI_{j,n}\}$. Since TA is noetherian, this collection has a maximal element $TI_{i,q}$. Then let each sequence $TI_{j,n}$ stabilize at n_j for each $j < i$. Set $m = \max (p, n_1, ..., n_{p-1})$. Now if $r, s > m$, we have $TI_{r,n} = TI_{s,n}$ for all n, and therefore $T'I_r = T'I_s$, which proves that the original sequence $\{T'I_j\}$ stabilizes as desired.

THEOREM 3.3. *Let $\mathscr{P}$ be a set of prime ideals of A which is saturated. Let B be an A-algebra such that TB is a noetherian object in $\mathscr{A}/\mathscr{C}$. Let $\mathscr{P}'$ be the set of prime ideals $\mathfrak{P}$ of B such that $\mathfrak{P} \cap A$ (the inverse image of $\mathfrak{P}$ in A under the algebra map $A \xrightarrow{\sigma} B$) is in $\mathscr{P}$. Then B is $\mathscr{P}'$-noetherian.*

Proof. Let $\mathscr{A}$ be the category of A-modules, $\mathscr{C}$ the subcategory of $\mathscr{A}$ associated to $\mathscr{P}$, and T the functor to the quotient category. Let $\mathscr{B}$, $\mathscr{D}$, and U be the corresponding data for B and $\mathscr{P}'$.

Suppose that $\{J_i\}$ is an ascending sequence of ideals of B. Then there is a k such that $TJ_k = TJ_m$ for all $m \geqslant k$. Let $\mathfrak{P}$ be in $\mathscr{P}'$ and set $\mathfrak{p} = \mathfrak{P} \cap A$ in $\mathscr{P}$. Then if $x \in J_m$ $(m \geqslant k)$, there is an $s \notin \mathfrak{p}$ such that $sx \in J_k$. Since $\sigma(s) \notin \mathfrak{P}$, we have $UJ_k = UJ_m$ for all $m \geqslant k$.

COROLLARY 3.4. *If B is a finitely generated A-module, then B is $\mathscr{P}'$-noetherian in the notation of* 3.3.

4. Primary Decomposition

We will establish a primary decomposition valid for A-modules M for which TM is noetherian. To express the theory along the usual lines we will work inside $\mathscr{A}$, rather than $\mathscr{A}/\mathscr{C}$; this makes elements available to us. We will follow the treatment given in ([*1*], Chapitre IV), establishing the critical propositions, and simply stating those whose proofs follow from the prior porpositions by obvious changes of wording in ([*1*], Chapitre IV). We will not attempt to carry over everything, but will establish the minimum body of material necessary for our immediate needs.

Let $\mathscr{P}$ be a saturated set of prime ideals of the ring A. Let $\mathscr{C}$ denote the associated subcategory of the category $\mathscr{A}$ of A-modules, $T : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ the associated functor and $S : \mathscr{A}/\mathscr{C} \to \mathscr{A}$ its adjoint. Let $F = ST$. Since T is exact and S is left exact, therefore F is left exact. For each A-module M let $\mathrm{Ass}_{\mathscr{P}}M$ denote those prime ideals $\mathfrak{p}$ in $\mathscr{P}$ such that FM contains a submodule isomorphic to $FA/F\mathfrak{p}$.

Lemma 4.1. $\mathrm{Ass}_{\mathscr{P}}M = \mathrm{Ass}_{\mathscr{P}}FM$.

Proof. Since TS is naturally equivalent to the identity functor, we have $FFM = STSTM = S(TS)\,TM$ isomorphic to $STM = FM$.

We now introduce a useful notational convention. Let M be in $\mathscr{A}$, and let M' be the maximal subobject of M in $\mathscr{C}$. Let $\bar{M}$ denote M/M'. Note that the action of A on $\bar{M}$ (and hence on FM) is the same as the action of $\bar{A}$ on $\bar{M}$. Also note that if $\mathfrak{p} \in \mathscr{P}$, then $(A/\mathfrak{p})^- = A/\mathfrak{p}$.

Lemma 4.2. $\mathrm{Ass}_{\mathscr{P}}M = \mathrm{Ass}_{\mathscr{P}}\bar{M}$.

Proof. $FM = F\bar{M}$.

Lemma 4.3. *Let* $\mathfrak{p} \in \mathscr{P}$. *Then* $\mathrm{Ass}_{\mathscr{P}}(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

Proof. Let $x \neq 0$ be in $F(A/\mathfrak{p})$. We will show that $\mathrm{Ann}_{FA}x = F\mathfrak{p}$. Since $x \in F(A/\mathfrak{p})$, there is an $s \notin \mathfrak{p}$ such that $sx \in A/\mathfrak{p}$. We claim $sx \neq 0$. For given any prime ideal $\mathfrak{q} \in \mathscr{P}$, there is a $t \notin \mathfrak{q}$ such that $tx \in A/\mathfrak{p}$. If $sx = 0$, then $s(tx) = 0$, so $tx = 0$ and hence $x = 0$.

Now let $a \in FA$ be such that $ax = 0$. Choose $s \notin \mathfrak{p}$ such that $sx \in A/\mathfrak{p}$ and for each $\mathfrak{q} \in \mathscr{P}$, choose $t \notin \mathfrak{q}$ such that $ta \in \bar{A}$. Then $(ta)(sx) = 0$ in $A/\mathfrak{p}$ and since $sx \neq 0$, we get $ta = 0$ in $A/\mathfrak{p}$. Therefore $ta \in \bar{\mathfrak{p}}$, and so $a \in F\mathfrak{p}$. This shows that $\mathrm{Ann}_{FA}x \subseteq F\mathfrak{p}$. If $v \in F\mathfrak{p}$, then for any $\mathfrak{q} \in \mathscr{P}$ choose s', $s'' \notin \mathfrak{q}$ such that $s'v \in \bar{\mathfrak{p}}$ and $s''x \in A/\mathfrak{p}$. Then $(s'v)(s''x) = 0$, so $(s's'')(vx) = 0$ and $s's'' \notin \mathfrak{q}$, so $vx = 0$.

LEMMA 4.4. *Let N be a submodule of M. Then* $\mathrm{Ass}_{\mathscr{P}}(M/N) = \mathrm{Ass}_{\mathscr{P}}(FM/FN)$.

Proof. It is clear that the maximal subobject of FM/FN in $\mathscr{C}$ is 0, and also that the kernel of $M/N \to FM/FN$ is the maximal subobject of M/N in $\mathscr{C}$. Since if $u \in FM$ and $\mathfrak{q} \in \mathscr{P}$, there is an $s \notin \mathfrak{q}$ such that $su \in M$, we have $FM/FN \subseteq F(M/N)$.

COROLLARY 4.5. *Let $\mathfrak{p} \in \mathscr{P}$, then* $\mathrm{Ass}_{\mathscr{P}}(FA/F\mathfrak{p}) = \{\mathfrak{p}\}$.

LEMMA 4.6. *Let $M = \bigcup M_i$. Then* $\mathrm{Ass}_{\mathscr{P}} M = \bigcup \mathrm{Ass}_{\mathscr{P}} M_i$.

Proof. That the right hand side is contained in the left hand side is clear. Now let $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{P}} M$. There is then an injection of $A/\mathfrak{p}$ into $\bar{M}$. Since $\bar{M} = \bigcup \bar{M}_i$, there is an injection of $A/\mathfrak{p}$ into $\bar{M}_i$ for some i. Thus $\mathfrak{p}$ is in the right hand side.

LEMMA 4.7. *Let M be an A-module. Every ideal of FA of the type $\mathrm{Ann}_{FA}\, x$ where x runs through the set of non-zero elements of FM is of the form FI for some ideal I of A. I is a prime ideal in $\mathscr{P}$ if $\mathrm{Ann}_{FA}\, x$ is chosen to be maximal*

Proof. Let $\mathfrak{a} = \mathrm{Ann}_{FA}\, x$, set $\bar{I} = \mathfrak{a} \cap \bar{A}$, and let I be the inverse image of $\bar{I}$ in A. Then $I = \{a \in A : ax = 0\}$. If $b \in \mathfrak{a}$, then $bx = 0$. Given a prime ideal $\mathfrak{p}$ in $\mathscr{P}$, choose $s \notin \mathfrak{p}$ such that $sb \in \bar{A}$. Surely $(sb)\, x = 0$, so $b \in FI$. Thus $\mathfrak{a} \subseteq FI$.

Now let $c \in FI$. If $\mathfrak{p} \in \mathscr{P}$, then there is an $s \notin \mathfrak{p}$ such that $sc \in \bar{I}$, therefore $scx = 0$. Since this holds for all $\mathfrak{p} \in \mathscr{P}$, and we have $cx = 0$ (the maximal subobject of FM in $\mathscr{C}$ is 0) we conclude that $c \in \mathfrak{a}$. So we have the opposite inclusion.

It is easy to see that if $\mathfrak{a}$ is chosen maximal among ideals of FA of the type $\mathrm{Ann}_{FA}\, x$ ($x \in FM$, $x \neq 0$), then $\mathfrak{a}$ is a prime ideal of FA. Therefore I is a prime ideal of A in $\mathscr{P}$.

COROLLARY 4.8. *Let A be $\mathscr{P}$-noetherian and let M be an A-module. Then* $\mathrm{Ass}_{\mathscr{P}} M = \emptyset$ *if, and only if,* $M \in \mathscr{C}$.

Proof. If $M \in \mathscr{C}$, then $FM = 0$ and clearly $\mathrm{Ass}_{\mathscr{P}} M = \emptyset$. If $FM \neq 0$, consider the ideals of the type $\mathrm{Ann}_{FA}\, x$ for $0 \neq x \in FM$. Such ideals are of the type FI for I an ideal of A and we may choose I to be a maximal such ideal. An application of Lemma 4.7 concludes the demonstration.

COROLLARY 4.9. *Let A be $\mathscr{P}$-noetherian, M an A-module, and $a \in A$. The multiplication $FM \xrightarrow{a} FM$ is injective if, and only if, a does not belong to a prime ideal in* $\mathrm{Ass}_{\mathscr{P}} M$.

Proof. If $a \in \mathfrak{p}$, where $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{P}} M$, then $F\mathfrak{p} = \mathrm{Ann}_{FA}\, x$ for some non-zero x in FM, so $ax = 0$ and multiplication by a is not injective. On the other hand if $as = 0$ for some $x \neq 0$ in FM, then let $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{P}}(FAx)$. Then $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{P}} M$, and clearly $a \in \mathfrak{p}$.

PROPOSITION 4.10. *Let N be a submodule of the A-module M. Then*

$$\mathrm{Ass}_{\mathscr{P}} N \subseteq \mathrm{Ass}_{\mathscr{P}} M \subseteq \mathrm{Ass}_{\mathscr{P}} N \cup \mathrm{Ass}_{\mathscr{P}}(M/N).$$

Proof. The first containment is evident. Now let $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{P}} M$. Then FM contains a submodule E isomorphic to $FA/F\mathfrak{p}$. Set $U = FN \cap E$. If $U \neq 0$, then $\{\mathfrak{p}\} = \mathrm{Ass}_{\mathscr{P}} U \subseteq \mathrm{Ass}_{\mathscr{P}} FN = \mathrm{Ass}_{\mathscr{P}} N$. If $U = 0$, then U is embedded in FM/FN, and so $\{\mathfrak{p}\} = \mathrm{Ass}_{\mathscr{P}} U \subseteq \mathrm{Ass}_{\mathscr{P}}(FM/FN) = \mathrm{Ass}_{\mathscr{P}} M/N$.

The following two corollaries follow as in ([*1*], Chapter IV).

COROLLARY 4.11. *If $M = \coprod M_i$, then $\mathrm{Ass}_{\mathscr{P}} M = \bigcup \mathrm{Ass}_{\mathscr{P}} M_i$.*

COROLLARY 4.12. *Let $\{Q_i\}$ be a finite family of submodules of M such that* $\bigcap Q_i = 0$. Then $\mathrm{Ass}_{\mathscr{P}} M \subseteq \bigcup \mathrm{Ass}_{\mathscr{P}}(M/Q_i)$.

We make explicit the following

LEMMA 4.13. *If $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{P}} M$, then there is a submodule N of M such that* $\mathrm{Ass}_{\mathscr{P}} N = \{\mathfrak{p}\}$.

Proof. Let $f : A/\mathfrak{p} \to \bar{M}$ be injective and let $N = \varphi^{-1}(f(A/\mathfrak{p}))$ where φ is the natural epimorphism of M to $\bar{M}$. Then, if M' denotes the maximal subobject of M in $\mathscr{C}$, we have the exact sequence $0 \to M' \to N \to A/\mathfrak{p} \to 0$, and so $\mathrm{Ass}_{\mathscr{P}} N \subseteq \mathrm{Ass}_{\mathscr{P}}(A/p) = \{\mathfrak{p}\}$.

Now follows formally

PROPOSITION 4.14. *Let Φ be a subset of $\mathrm{Ass}_{\mathscr{P}} M$. There is a submodule N of M such that $\mathrm{Ass}_{\mathscr{P}} N = \mathrm{Ass}_{\mathscr{P}} M - \Phi$, and $\mathrm{Ass}_{\mathscr{P}}(M/N) = \Phi$.*

THEOREM 4.15. *Let A be $\mathscr{P}$-noetherian and let M be an A-module such that TM is noetherian. Then there is a chain of submodules $\{M_i\}$ $0 \leqslant i \leqslant n$ of M such that $M = M_0$, and for $0 \leqslant i < n$, $(M_i/M_{i+1})^-$ is isomorphic to $A/\mathfrak{p}_i$ for $\mathfrak{p}_i \in \mathscr{P}$.*

Proof. Let S be the collection of submodules of M which satisfy the conditions of the theorem. Choose N' among them such that TN' is maximal in TM. By Zorn's lemma, we may then choose N maximal in M such that $TN = TN'$. Then $M/N = (M/N)^-$. If M/N is in $\mathscr{C}$ we are done. If M/N is not in $\mathscr{C}$ choose a submodule L/N of M/N which is isomorphic to $A/\mathfrak{p}$ for some $\mathfrak{p} \in \mathscr{P}$. Then $\mathrm{Ass}_{\mathscr{P}} L \supseteq \mathrm{Ass}_{\mathscr{P}} N \cup \{\mathfrak{p}\}$, a contradiction.

COROLLARY 4.16. *If TM is noetherian, then* $\mathrm{Ass}_{\mathscr{P}} M$ *is finite.*

PROPOSITION 4.17. *Let A be $\mathscr{P}$-noetherian and suppose that TM is noetherian and that* $\mathrm{Ass}_{\mathscr{P}} M = \{\mathfrak{p}\}$. *Then if $a \in \mathfrak{p}$, the multiplication* $FM \xrightarrow{a} FM$ *is nilpotent.*

Proof. Consider the chain of submodules $0 \subseteq \mathrm{Ann}_{FM}\, a \subseteq \mathrm{Ann}_{FM}\, a^2 \subseteq \cdots$ of FM. It is easy to check that these are $\mathscr{C}$-closed, and therefore there is an n such that $\mathrm{Ann}_{FM} a^n = \mathrm{Ann}_{FM} a^{n+1}$. Suppose $a^n FM \neq 0$. Then $\{\mathfrak{p}\} = \mathrm{Ass}_{\mathscr{P}}(a^n FM)$, but the multiplication $a^n FM \xrightarrow{a} a^n FM$ is not injective. Therefore there is an $x \in FM$ such that $a^n x \neq 0$, but $a^{n+1} x = 0$. This contradicts $\mathrm{Ann}_{FM}\, a^n = \mathrm{Ann}_{FM}\, a^{n+1}$.

COROLLARY 4.18. *Let A and M be $\mathscr{P}$-noetherian. Then* $\mathrm{Ass}_{\mathscr{P}} M$ *is a singleton if, and only if, M is not in $\mathscr{C}$ and every multiplication of FM by an element of A is either injective or nilpotent.*

We now are prepared to define $\mathfrak{p}$-primary modules for $\mathfrak{p} \in \mathscr{P}$. If A and M are $\mathscr{P}$-noetherian and $\mathrm{Ass}_{\mathscr{P}} M/N = \{\mathfrak{p}\}$, then we say that N is $\mathfrak{p}$-primary with respect to M.

THEOREM 4.19. *Let A and M be $\mathscr{P}$-noetherian and let N be a submodule of M. Then there are submodules Q_i of M, each primary with respect to some prime ideal $\mathfrak{p}_i$ in* $\mathrm{Ass}_{\mathscr{P}} M/N$, *such that $N \subseteq E = \bigcap Q_i$ and $E/N \in \mathscr{C}$.*

Proof. We may suppose $N = 0$. Choose for any $\mathfrak{p}_i \in \mathrm{Ass}_{\mathscr{P}} M$ a submodule Q_i such that $\mathrm{Ass}_{\mathscr{P}} M/Q_i = \{\mathfrak{p}_i\}$, and $\mathrm{Ass}_{\mathscr{P}} Q_i = \mathrm{Ass}_{\mathscr{P}} M - \{\mathfrak{p}_i\}$. Set $E = \bigcap Q_i$. For each i, $E \subseteq Q_i$, so $\mathrm{Ass}_{\mathscr{P}} E$ is contained in $\mathrm{Ass}_{\mathscr{P}} Q_i$ for each i, and hence $\mathrm{Ass}_{\mathscr{P}} E = \emptyset$ and therefore $E \in \mathscr{C}$.

COROLLARY 4.20. *Let A and M be $\mathscr{P}$-noetherian and let N be a submodule of M. Then $FN = \bigcap FQ_i$ and each FQ_i is $\mathfrak{p}_i$-primary in FM (in the notation of the above theorem).*

Proof. Apply the (left exact) functor F to $E = \bigcap Q_i$ and note that $FN = FE$.

We now derive some consequences for ideals of A which will be needed in the sequel.

LEMMA 4.21. *Let I be an ideal of the $\mathscr{P}$-noetherian ring A. Then* $\mathfrak{p} \in \mathrm{Ass}_{\mathscr{P}} A/I$ *implies* $\mathfrak{p} \supseteq I$.

Proof. Let J be the kernel of the composition $A \to A/I \to (A/I)^{-}$. Then $A/J = (A/I)^{-}$ and surely $J \supseteq I$. By assumption there is an injection $A/\mathfrak{p} \to A/J$ which implies that $\mathfrak{p} \supseteq J \supseteq I$.

PROPOSITION 4.22. *Let A be $\mathscr{P}$-noetherian, I an ideal of A. Suppose that $\mathfrak{p} \in \mathscr{P}$ and that $\mathfrak{p}$ is a minimal prime ideal containing I. Then $\mathfrak{p} \in \operatorname{Ass}_{\mathscr{P}} A/I$.*

Proof. Let $FI = \bigcap FV_j$ where FV_j is primary in FA for $\mathfrak{q}_j$ in $\mathscr{P}$. Then $F\mathfrak{p} \supseteq FI \supseteq \prod FV_j$. Since $F\mathfrak{p}$ is a prime ideal of FA we have $F\mathfrak{p} \supseteq FV_i$ for some i. Now let $q \in \mathfrak{q}_i$. We have that $FA/FV_i \xrightarrow{q} FA/FV_i$ is a nilpotent mapping and therefore $q^n \in FV_i \subseteq F\mathfrak{p}$ for some n. This gives $q \in F\mathfrak{p} \cap A = \mathfrak{p}$. Thus $\mathfrak{p} \supseteq \mathfrak{q}_i \supseteq I$ and since $\mathfrak{p}$ is minimal over I, $\mathfrak{p} = \mathfrak{q}_i$.

THEOREM 4.23. *Let A be $\mathscr{P}$-noetherian. Then the set of prime ideals of A in $\mathscr{P}$ minimal over I is finite.*

Proof. Such a $\mathfrak{p}$ must be in $\operatorname{Ass}_{\mathscr{P}} A/I$ and this is finite.

PROPOSITION 4.24. *Let A be $\mathscr{P}$-noetherian, I an ideal of A with $\operatorname{Ass}_{\mathscr{P}}(A/I) = \{\mathfrak{p}\}$. Then $IA_{\mathfrak{q}}$ is a primary ideal with radical $\mathfrak{p}A_{\mathfrak{q}}$ or is $A_{\mathfrak{q}}$ for all prime ideals $\mathfrak{q} \in \mathscr{P}$.*

5. KRULL DOMAINS

From Example 2.5, we have that if A is a Krull domain, then A is $\mathscr{P}_1$-noetherian, where $\mathscr{P}_1$ denotes the set of prime ideals of A of height one or less.

In this section we want to study the connection of the functor F with the further hypothesis that A be a Krull domain.

If A is any domain with field of quotients K, then K is the injective envelope of A. If $\mathscr{S}$ is any set of prime ideals of A (which we assume to be saturated as always), then FA relative to $\mathscr{S}$ is isomorphic to $\{x \in K : \forall\, \mathfrak{p} \in \mathscr{S}, \exists\, s \notin \mathfrak{p}$ such that $sx \in A\} = \bigcap A_{\mathfrak{p}}$. We then have

PROPOSITION 5.1. *A is a Krull domain if, and only if, the following three conditions are satisfied:*

(i) A is $\mathscr{P}_1$-noetherian and an integral domain.

(ii) $FA = A$ (F relative to $\mathscr{P}_1$).

(iii) $A_{\mathfrak{p}}$ is regular for each $\mathfrak{p} \in \mathscr{P}_1$.

Let $\mathbf{P}$ be the prime ideals in $\mathscr{P}_1$ of height one.

If M is a torsion free module over a Krull domain A, then since $K \otimes_A M$ is the injective envelope of M, $FM = \{X \in K \otimes_A M : \forall\, \mathfrak{p} \in \mathscr{P}_1, \exists\, s \notin \mathfrak{p}$ such that $sX \in M\} = \bigcap M_{\mathfrak{p}}$.

For future reference, we include at this point several results concerning lattices over a Krull domain. The essential points are found in ([*1*], Chapitre VII, Section 4), and as in the previous section, proofs are given here only where the techniques deviate substantially from those found there.

We review the definition of an A-lattice. Let V be a finite dimensional vector space over the field K. An *A-lattice* M in V is an A-submodule of V which contains a K-basis of V (and thus a free A-module) and is contained in a finitely generated (and hence a free) A-submodule of V.

PROPOSITION 5.2. *An A-submodule M of V is an A-lattice if, and only if, M is $\mathscr{P}_1$-finitely generated and spans V over K.*

Proof. If M is an A-lattice in V, then there is a free module A^n in V containing M. Then TM is contained in TA^n, which is a noetherian object. Hence M is $\mathscr{P}_1$-finitely generated. Conversely if M is $\mathscr{P}_1$-finitely generated, there is a set of elements of M, $\{x_1, ..., x_r\}$, such that $TM = T(\sum Ax_i)$. This set must contain a basis of V, say $y_1, ..., y_n$, and there is an $a \in A$, $a \neq 0$, such that $ax_i \in \sum Ay_j$ for all i, and hence $\sum Ax_i \subseteq \sum A(a^{-1}y_j) = F$. Since $TM = T(\sum Ax_i)$, we know that $M_\mathfrak{p} = (\sum Ax_i)_\mathfrak{p}$ for all $\mathfrak{p} \in \mathscr{P}_1$. Now $M \subseteq \bigcap M_\mathfrak{p} \subseteq \bigcap F_\mathfrak{p}$. We will be done when we show that $\bigcap F_\mathfrak{p} = F$.

Clearly $F \subseteq \bigcap F_\mathfrak{p}$. On the other hand, if $x \in \bigcap F_\mathfrak{p}$, write x (uniquely) as $x = u_1y_1 + \cdots + u_ny_n$ with $u_i \in K$. Then $x \in F_\mathfrak{p}$ means that each $u_i \in A_\mathfrak{p}$ for each $\mathfrak{p}$, and hence $u_i \in A$. The proof is now complete.

The next proposition is in ([*1*], Chapitre VII, Section 4, no. 1).

PROPOSITION 5.3. (i) *Let M_1, M_2 be two A-lattices in V. Then $M_1 \cap M_2$ is an A-lattice in V.*

(ii) *Let W be a K-subspace of V and M be an A-lattice in V. Then $W \cap M$ is an A-lattice in W.*

(iii) *Let $V_1, ..., V_r$, U be finite dimensional vector spaces over K and $f: V_1 \times \cdots \times V_r \to U$ a multilinear form with* $\operatorname{Im} f$ *generating U. Suppose M_i is an A-lattice in V_i for each i. Then $F(M_1 \times \cdots \times M_r)$ is an A-lattice in U.*

(iv) *Let V, W be finite dimensional K vector spaces, M an A-lattice in V, N an A-lattice in W. Then $N : M = \{f \in \operatorname{Hom}_K(V, W) : f(M) \subseteq N\}$ is an A-lattice in* $\operatorname{Hom}_K(V, W)$.

According to ([*1*], Chapitre VII, Section 4, Remarque 3) we can identify $N : M$ with $\operatorname{Hom}_A(M, N)$, and thus $A : M$ with $M^* = \operatorname{Hom}_A(M, A)$.

COROLLARY 5.4. *Let U, V, W be finite dimensional vector spaces over K and $f: U \times V \to W$ a bilinear form nondegenerate in U. Let M be an A-lattice in V and N an A-lattice in W. Then $N :_f M = \{u \in U : f(u, M) \subseteq N\}$ is an A-lattice in V.*

Since it is not generally the case that $\mathrm{Hom}_A(M, N)_S = \mathrm{Hom}_{A_S}(M_S, N_S)$ for any A-modules M and N and multiplicatively closed subset S of A (M may fail to be finitely presented) and since we would like this to hold in some special cases, a modification of the treatment in ([1], Chapitre VII, Section 4, no 2) is necessary. This amounts to a rearrangement of the statements.

We say an A-lattice M in V is *divisorial* if it is $\mathscr{C}$-closed. This is the same as saying $M = FM = \bigcap M_{\mathfrak{p}}$ for $ht\mathfrak{p} = 1$. We have seen in the proof of Proposition 5.2, that a free A-module in V is divisorial. This establishes (i) of

THEOREM 5.5 *Let V be a finite dimensional K vector space. Let L be a free A-module of V.*

(i) *L is a divisorial A-lattice in V.*

(ii) *Let $L' \subseteq L$ be a free A-submodule of V with $rk_A L' = rk_K V$. Then $L'_{\mathfrak{p}} = L_{\mathfrak{p}}$ for all but a finite number of prime ideals $\mathfrak{p} \in \mathbf{P}$.*

(iii) *Suppose, for each $\mathfrak{p} \in \mathbf{P}$, we are given an $A_{\mathfrak{p}}$-lattice $N(\mathfrak{p})$ in V such that $N(\mathfrak{p}) = L_{\mathfrak{p}}$ for all but a finite number of $\mathfrak{p} \in \mathbf{P}$. Then $M = \bigcap N(\mathfrak{p})$ is an A-lattice in V. Furthermore $M_{\mathfrak{p}} = N(\mathfrak{p})$ for each $\mathfrak{p} \in \mathbf{P}$, and hence M is a divisorial A-lattice. M is the only divisorial A-lattice N with $N_{\mathfrak{p}} = N(\mathfrak{p})$ for all $\mathfrak{p} \in \mathbf{P}$.*

(iv) *If N is an A-lattice in V, then $N_{\mathfrak{p}} = M_{\mathfrak{p}}$ for all but a finite number of $\mathfrak{p} \in \mathbf{P}$.*

Proof. (i) has been established and (iv) follows from it. To prove (ii), let (a_{ij}) be the element of $\mathbf{M}_n(A)$ giving a fixed basis of L' in terms of a fixed basis of L. Let $d = \det(a_{ij})$. d is contained in only a finite number of prime ideals $\mathfrak{p} \in \mathbf{P}$ and $d \neq 0$. If $d \notin \mathfrak{p}$, then $L'_{\mathfrak{p}} = L_{\mathfrak{p}}$.

The proof of (iii) follows as in ([1], Chapitre VII, Section 4), and is omitted.

Remark. An A-lattice in K is a fractionary A-ideal of K. It is divisorial in our sense if, and only if, it is divisorial in the usual sense.

PROPOSITION 5.6. (i) *If M_1 and M_2 are divisorial A-lattices in V, then $M_1 \cap M_2$ is a divisorial A-lattice in V.*

(ii) *If W is a K-subspace of V, and M is a divisorial A-lattice in V, then $M \cap W$ is a divisorial A-lattice in W.*

(iii) *If V and W are finite dimensional K-vector spaces, M an A-lattice in V, N a divisorial A-lattice in W, then $N : M$ is a divisorial A-lattice in* $\mathrm{Hom}_K(V, W)$.

(iv) *Let U, V, W be finite dimensional K-vector spaces, $f : U \times V \to W$ a bilinear form nondegenerate in U. Let M be an A-lattice in V, N a divisorial A-lattice in W. Then $N :_f M$ is a divisorial A-lattice in U.*

Proof. (i) and (ii) are clear. To prove (iii) we note that $N = \bigcap N_{\mathfrak{p}}$. Let $f \in N : M$, then $f(M_{\mathfrak{p}}) \subseteq N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathbf{P}$, and hence $(N : M)_{\mathfrak{p}} \subseteq N_{\mathfrak{p}} : M_{\mathfrak{p}}$. Suppose $f \in \bigcap N_{\mathfrak{p}} : M_{\mathfrak{p}}$. Then for each $x \in M \subseteq M_{\mathfrak{p}}$ we have $f(x) \in N_{\mathfrak{p}}$, and hence $f(x) \in \bigcap N_{\mathfrak{p}} = N$. Thus we conclude that $N : M = \bigcap N_{\mathfrak{p}} : M_{\mathfrak{p}}$. Thus $N : M$ is divisorial by Proposition 5.4, and we conclude as well that $(N : M)_{\mathfrak{p}} = N_{\mathfrak{p}} : M_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathbf{P}$. The remainder of the proposition follows as in ([*1*], Chapter VII, Section 4).

The fact noted at the end of the above proof is useful enough to state independently.

COROLLARY 5.7. *Let V and W be finite dimensional K-spaces. If M is an A-lattice in V and N a divisorial A-lattice in W, then $(N : M)_{\mathfrak{p}} = N_{\mathfrak{p}} : M_{\mathfrak{p}}$, and so, in particular,* $\operatorname{Hom}_A(M, N)_{\mathfrak{p}} = \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.

COROLLARY 5.8. *Let M be an A-lattice in V. Then $M^* = \operatorname{Hom}_A(M, A)$ is a divisorial A-lattice in K and $(M_{\mathfrak{p}})^* = (M^*)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathbf{P}$. Furthermore M is divisorial if and only if $M = M^{**}$.*

6. i-NOETHERIAN RINGS

Let A be a commutative ring and $\mathscr{P}_i$ the set of prime ideals of A of height $\leqslant i$ in A. We say that A is *i-noetherian* if A is noetherian with respect to the set of prime ideals $\mathscr{P}_i$. The next few statements are immediate corollaries of the results in Section 3.

THEOREM 6.1. *A is j-noetherian for any $j \leqslant i$.*

THEOREM 6.2. *If A is i-noetherian, then A_S is i-noetherian for any multiplicatively closed subset S of A.*

THEOREM 6.3. *If A is i-noetherian, the $A[X]$ is i-noetherian.*

Proof. If $\mathfrak{q}$ is a prime ideal of $A[X]$ of height less than or equal to i, then $\mathfrak{p} = \mathfrak{q} \cap A$ has height $\leqslant i$.

COROLLARY 6.4. *If A is i-noetherian, then $A[X_1, \ldots, X_n]$ is i-noetherian.*

We can do substantially better than this however. For we have

THEOREM 6.5. *Let $\{X_a\}$ be any set of indeterminants. If A is i-noetherian, then $A[\{X_a\}]$ is i-noetherian.*

Proof. It suffices to show that any prime ideal $\mathfrak{P}$ of $B = A[\{X_a\}]$ is $\mathscr{P}_i$-finitely generated. If $ht\mathfrak{P} < \infty$, then it is easy to see that $\mathfrak{P}$ is the extension of $\mathfrak{p} = \mathfrak{P} \cap A[X_1, ..., X_n]$ for some finite subset $\{X_i\}$ of $\{X_a\}$. Let $w_1, ..., w_r$ be a finite set of elements of $\mathfrak{p}$ such that $T'\mathfrak{p} = T'(\sum B'w_i)$, where T' is the functor associated with the set $\mathscr{P}_i(B')$ (where $B' = A[X_1, ..., X_n]$). Let $\mathfrak{Q}$ be any prime ideal of B of height $\leqslant i$. Then $ht(\mathfrak{Q} \cap B') \leqslant i$. Let $x \in \mathfrak{P}$, then $x = \sum u_i p_i$ where $p_i \in \mathfrak{p}$ and $u_i \in B$. Since there is an $s_i \in B' - (\mathfrak{Q} \cap B')$, such that $s_i p_i \in \sum B'w_i$, we get $(\prod s_i)\, x \in \sum Bw_i$. Thus $T\mathfrak{P} = T(\sum Bw_i)$.

Now suppose that $ht\mathfrak{P} < i$. Since the minimal prime ideals of (0) in B are extensions of the minimal prime ideals of (0) in A, these are finite in number (Theorem 4.24). Choose $a_1 \in \mathfrak{P}$ and not in any of these minimal prime ideals. a_1 involves only a finite number of variables, say, $X_1, ..., X_k$ and therefore there are only a finite number of prime ideals of $A[X_1, ..., X_k]$ of height 1 containing a_1 and the prime ideals of B of height 1 containing a_1 are extensions of these. Continue this process to get $a_1, ..., a_i \in \mathfrak{P}$ such that there are only a finite number of prime ideals $\mathfrak{Q}_1, ..., \mathfrak{Q}_t$ of height i in B containing $\sum Ba_i$. Then choose $a_{i+1} \in \mathfrak{P} - \bigcup \mathfrak{Q}_j$ and set $I = \sum Ba_i$. If $\mathfrak{Q}$ is a prime ideal of B of height at most i, then $\mathfrak{Q} \not\supseteq I$, so $T\mathfrak{P} = TI$.

COROLLARY 6.6. *If A is a noetherian ring, then $A[\{X_a\}]$ is i-noetherian for each $i \geqslant 0$.*

7. GENERALIZED CLASS GROUPS

In this section we show how to define, for an n-noetherian ring A, a sequence of groups $W_0(A)$, $W_1(A)$,..., $W_n(A)$, which are the same groups as those defined in [2] for a noetherian ring A. Since most of the general theorems concerning these groups follow as in [2], only an outline of the details is presented here.

Some notation is necessary. Let $\mathscr{A}$ be an abelian category. $K^0(\mathscr{A})$ denotes the Grothendieck group of the Serre subcategory of $\mathscr{A}$ consisting of those objects of $\mathscr{A}$ which are noetherian. We let $\mathscr{A}^{\#}$ denote this subcategory of $\mathscr{A}$. When A is a commutative ring, let $\mathscr{M}_{i+1}$ denote the Serre subcategory of the category $\mathscr{M}$ of A-modules associated with the set of prime ideals $\mathscr{P}_i$. Let $T_i : \mathscr{M} \to \mathscr{M}/\mathscr{M}_{i+1}$ denote the associated functor to the quotient category. We note that $\mathscr{M}_i \supseteq \mathscr{M}_{i+1}$ for all i, and hence for $j > i$ we have induced an embedding $\mathscr{M}_{i+1}/\mathscr{M}_{j+1} \to \mathscr{M}/\mathscr{M}_{j+1}$, with associated quotient functor $T_{ij} : \mathscr{M}/\mathscr{M}_{j+1} \to \mathscr{M}/\mathscr{M}_{i+1}$.

LEMMA 7.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ an exact functor. Then F induces an exact functor $F^{\#} : \mathscr{A}^{\#} \to \mathscr{B}^{\#}$ by restriction if either of the two conditions below are satisfied.*

(i) *F is an embedding, such that if $N \in \mathscr{B}$ is a subobject (quotient object) of FM, $M \in \mathscr{A}$, then $N = FM'$ some $M' \in \mathscr{A}$ a subobject (quotient object) of M.*

(*ii*) *F is a functor from $\mathscr{A}$ to a quotient category $\mathscr{B}$.*

The proof is clear and is omitted. The cases of most interest to us are

(a) F is an exact embedding of a Serre subcategory $\mathscr{A}$ of $\mathscr{B}$. In this case $\mathscr{A}^{\#} = \mathscr{A} \cap \mathscr{B}^{\#}$.

(b) $\mathscr{A}$ admits a noetherian generator N, and F is a functor to a quotient category $\mathscr{B}$ such that FN is a generator of $\mathscr{B}$. In this case $F^{\#}$ is onto objects.

PROPOSITION 7.2. *Let A be a commutative ring. For each i, $0 \leqslant i \leqslant n$, let $D_i(A)$ be the Grothendieck group of the subcategory of $\mathscr{M}_i/\mathscr{M}_{i+1}$ consisting of the objects of finite length. The assignment $M \to \sum_{ht\mathfrak{p}=i} l_{A_\mathfrak{p}}(M_\mathfrak{p})\langle\mathfrak{p}\rangle$ induces an isomorphism $D_i(A) \approx \coprod_{ht\mathfrak{p}=i} \mathbf{Z}_\mathfrak{p}$ where $\mathbf{Z}_\mathfrak{p} = \mathbf{Z}$ for each $\mathfrak{p}$.*

Proof. It is sufficient to calculate the Grothendieck group of the associated semi-simple subcategory generated by the simple objects of $\mathscr{M}_i/\mathscr{M}_{i+1}$. Hence it is necessary to know what the simple objects are. We claim that an object is simple if, and only if, it is $T_i(A/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of A, of height i. First suppose that T_iM is a subobject of $T_i(A/\mathfrak{p})$. Then there is a subobject M' of $A/\mathfrak{p}$ with $T_iM' = T_iM$, and hence an ideal $\mathfrak{a}$ of A with $\mathfrak{p} \subseteq \mathfrak{a} \subseteq A$ and $\mathfrak{a}/\mathfrak{p} = M'$. If $\mathfrak{q} \neq \mathfrak{p}$ is a prime ideal with $ht\mathfrak{q} = i$, then $\mathfrak{a}A_\mathfrak{q} = A_\mathfrak{q}$ since $\mathfrak{p}A_\mathfrak{q} = A_\mathfrak{q}$. If $\mathfrak{a} \neq \mathfrak{p}$, then also $\mathfrak{a}A_\mathfrak{p} = A_\mathfrak{p}$, so $T_i(A/\mathfrak{a}) = 0$ or $T_i(\mathfrak{a}/\mathfrak{p}) = T_i(A/\mathfrak{p})$. Thus $T_iM = 0$ or $T_iM = T_i(A/\mathfrak{p})$. Second suppose T_iM is a simple object. Since T_i preserves colimits, and A is a generator of $\mathscr{M}$, there is an $x \in M$ such that $T_i(Ax) \to T_i(M)$ is not zero. Let $\mathfrak{a} = \operatorname{Ann}_A x$. Then $f : A/\mathfrak{a} \to M$ is a monomorphism with $T_i f \neq 0$. Since T_iM is simple $T_i f$ is an isomorphism. Since $T_i(A/\mathfrak{a}) \neq 0$, there is prime ideal $\mathfrak{p}$ of A of height i such that $\mathfrak{a}A_\mathfrak{p} \neq A_\mathfrak{p}$ and hence $\mathfrak{a} \subseteq \mathfrak{p}$. Now $A/\mathfrak{a} \to A/\mathfrak{p} \to 0$ exact induces $T_i(A/\mathfrak{a}) \to T_i(A/\mathfrak{p}) \to 0$ exact. Since T_iM is simple and isomorphic to $T_i(A/\mathfrak{a})$, therefore there is a non-zero morphism $T_iM \to T_i(A/\mathfrak{p})$. Since both are simple, this is an isomorphism. The claim is now established and the proposition follows.

LEMMA 7.3. *If A is n-noetherian and $i \leqslant n$, then $(\mathscr{M}_i/\mathscr{M}_{i+i})^{\#}$ is the subcategory of objects of finite length in $\mathscr{M}_i/\mathscr{M}_{i+1}$.*

Proof. Let M in $\mathscr{M}_i$ be such that T_iM is in $(\mathscr{M}_i/\mathscr{M}_{i+1})^{\#}$. By Theorem 4.15, there is a chain of submodules $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq M_{k+1} = 0$, such that $T_i(M_j/M_{j+1}) = T_i(A/\mathfrak{p}_j)$ for $0 \leqslant j \leqslant k$ with $\mathfrak{p}_j \in \mathscr{P}_i$. Hence T_iM has a composition series of finite length.

COROLLARY 7.4. *If A is n-noetherian then, for $i \leqslant n$,*

$$K^0(\mathcal{M}_i/\mathcal{M}_{i+1}) = D_i(A) = \coprod_{ht\mathfrak{p}=i} \mathbf{Z}_\mathfrak{p}\,, \mathbf{Z}_\mathfrak{p} = \mathbf{Z}.$$

PROPOSITION 7.5. *Let $k_i(A)$ denote $K^1(\mathscr{T})$ where $\mathscr{T}$ denotes the category of objects of finite length in $\mathcal{M}_i/\mathcal{M}_{i+1}$. Then*

$$k_i(A) = \coprod_{ht\mathfrak{p}=i} U(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$$

where $U(B)$ denotes the group of units of the ring B.

Proof. Once again it is sufficient to consider the semisimple subcategory of $\mathscr{T}$. The computation follows as in ([2], Proposition 2.5 and Lemma 2.6).

COROLLARY 7.6. *If A is n-noetherian and $i \leqslant n$, then*

$$k_i(A) = K^1(\mathcal{M}_i/\mathcal{M}_{i+1}) = \coprod_{ht\mathfrak{p}=i} U(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}).$$

LEMMA 7.7. *Let A be n-noetherian. For each $j < i \leqslant n$, $\mathcal{M}_i/\mathcal{M}_{i+1}$ is a Serre subcategory of $\mathcal{M}_j/\mathcal{M}_{i+1}$, and the quotient category is naturally equivalent to $\mathcal{M}_j/\mathcal{M}_i$. This statement holds as well when the operation $(\ )^\#$ is applied to the three categories.*

Let A be an n-noetherian ring and i an integer with $1 \leqslant i \leqslant n$. In light of Lemma 7.7, we may consider the exact sequence of abelian groups $K^1(\mathcal{M}_{i-1}/\mathcal{M}_i) \xrightarrow{\delta} K^0(\mathcal{M}_i/\mathcal{M}_{i+1}) \xrightarrow{\lambda} K^0(\mathcal{M}_{i-1}/\mathcal{M}_{i+1}) \xrightarrow{\mu} K^0(\mathcal{M}_{i-1}/\mathcal{M}_i) \longrightarrow 0$ where the homomorphisms are those given in [2]. Define for each i the group $W_i(A)$ to be $\operatorname{Im}\lambda = \operatorname{Ker}\mu$ in $K^0(\mathcal{M}_{i-1}/\mathcal{M}_{i+1})$. Since $K^0(\mathcal{M}_{i-1}/\mathcal{M}_i)$ is free by Corollary 7.4, we conclude that $W_i(A)$ is a direct summand of $K^0(\mathcal{M}_{i-1}/\mathcal{M}_{i+1})$. We can also say that $W_i(A)$ is isomorphic to $D_i(A)$ modulo$_j$ the subgroup generated by the elements

$$\sum_{ht\mathfrak{p}=i} A_\mathfrak{p}/(\mathfrak{Q} + xA)_\mathfrak{p})\langle\mathfrak{p}\rangle$$

as $\mathfrak{Q}$ runs over the prime ideals of A of height $i - 1$ and $x \notin \mathfrak{Q}$. When A is a noetherian ring the groups defined here and the groups $W_i(A)$ defined in [2] are the same.

According to the description of $W_i(A)$ as a factor group of $D_i(A)$ we get immediately the

THEOREM 7.8. *Let A be a Krull domain. Then $D_1(A)$ is isomorphic to the ordinary group of divisors of A. $W_1(A)$ is isomorphic to the ordinary ideal class group $C(A)$ of A.*

Proof. The first statement follows from Corollary 7.4, the second from the fact that $W_1(A)$ is $D_1(A)$ modulo, the group of principal ideals as seen in the discussion above.

Remark. We have, in Theorem 7.8, obtained a module theoretical description of the ideal class group $C(A)$ of a Krull domain A in a manner similar to that obtained for a noetherian Krull domain found in ([*1*], Chapitre VII, Section 4, no. 7, Proposition 17).

In [*2*] we showed how, given a noetherian ring A and a flat noetherian A-algebra B, to determine in a natural manner a homomorphism $W_i(A) \to W_i(B)$ for each i. This does not handle the many cases where B is an A-algebra, not necessarily flat, but yet where a natural homomorphism of the class group does exist. One case, in particular, of this is found in ([*1*], Chapitre VII, Section 1, no. 10). If A is a Krull domain and B a Krull domain containing A; suppose the pair (A, B) satisfies the condition (PDE): If $\mathfrak{P}$ is a prime ideal of B of height one, then $ht(\mathfrak{P} \cap A) \leqslant 1$. Then there is a homomorphism of class groups $C(A) \to C(B)$ induced by the assignment $\mathfrak{p} \to \mathfrak{p}B$ for a prime ideal $\mathfrak{p}$ of height one in A. Note that the property implies for each prime ideal $\mathfrak{P}$ of height one in B that $B_{\mathfrak{P}}$ is a flat $A_{\mathfrak{p}}$-module.

We offer here a generalization of this which permits the defining in a natural manner homomorphisms $W_i(A) \to W_i(B)$ for some extension B of A.

Suppose B is an A-algebra. Consider the property

(F_m) If $\mathfrak{Q}$ is a prime ideal of B with $ht\mathfrak{Q} \leqslant m$
and $\mathfrak{p} = \mathfrak{Q} \cap A$ then $B_{\mathfrak{Q}}$ is a flat $A_{\mathfrak{p}}$-module.

If B satisfies (F_m) we say that B is an *m-flat* A-algebra.

THEOREM 7.9. *Suppose B is an A-algebra and that A and B are n-noetherian for some integer n. Suppose further that B is n-flat. If $\mathfrak{Q}$ is a prime ideal of B with $ht\mathfrak{Q} \leqslant n$, then $ht(\mathfrak{Q} \cap A) \leqslant ht\mathfrak{Q}$.*

Proof. Let $\mathfrak{p} = \mathfrak{Q} \cap A$. $B_{\mathfrak{Q}}$ is a noetherian ring and a faithfully flat $A_{\mathfrak{p}}$-module. Hence $A_{\mathfrak{p}}$ is a noetherian ring by ([*1*], Chapitre I, Section 3, no. 5). We are thus reduced to showing the following: If A and B are local rings and B is a faithfully flat A-algebra, then Krull dim $B \geqslant$ Krull dim A.

First, we show that if $a \in A$ is not in any minimal prime ideal of A, then aB is not contained in any minimal prime ideal of B. For suppose $\mathfrak{Q}$ is a minimal prime ideal of B and that $aB \subset A$. Then $a \in \mathfrak{p} = \mathfrak{Q} \cap A$. Hence $B_{\mathfrak{Q}}$ is a primary ring which is a faithfully flat $A_{\mathfrak{p}}$-module and Krull dim $A > 0$. The multiplication $B_{\mathfrak{Q}} \xrightarrow{a} B_{\mathfrak{Q}}$ is nilpotent since $B_{\mathfrak{Q}}$ is primary and $a \in \mathfrak{Q}$. Hence $a^r = 0$ for some integer r and the faithfulness of $B_{\mathfrak{Q}}$ over $A_{\mathfrak{p}}$. This is a contradiction.

We now prove Krull dim $A \leqslant$ Krull dim B by induction on Krull dim A. The case Krull dim $A = 0$ is trivial. If Krull dim $A > 0$, let $a \in A$ be such that a is not in any minimal prime ideal of A and is not a unit. Then B/aB is a faithfully flat A/aA-module, so we have, by the induction hypothesis

$$\begin{aligned}\text{Krull dim } A &= 1 + \text{Krull dim } A/aA \leqslant 1 + \text{Krull dim } B/aB \\ &= \text{Krull dim } B\end{aligned}$$

since aB is contained in no minimal prime ideal of B.

PROPOSITION 7.10. *Let B be an A-algebra with A and B n-noetherian. Suppose B is n-flat.*

(i) *If $r \leqslant n$, then $\mathscr{M}_{r+1} \otimes B \subseteq \mathscr{N}_{r+1}$.*

(ii) *$- \otimes_A B$ induces an exact functor $\mathscr{M}/\mathscr{M}_{r+1} \to \mathscr{N}/\mathscr{N}_{r+1}$ for each $r \leqslant n$.*

Here $\mathscr{N}$ denotes the category of B-modules.

Proof. Suppose M is an A-module such that $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of A in $\mathscr{P}_r$. Let $\mathfrak{Q}$ be a prime ideal of B with $ht\mathfrak{Q} \leqslant r$. Then $(M \otimes_A B) \otimes_B B_{\mathfrak{Q}} = (M \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{Q}}$ where $\mathfrak{p} = \mathfrak{Q} \cap A$. But this is zero. Hence the first claim is established.

Let $T' : \mathscr{N} \to \mathscr{N}/\mathscr{N}_{r+1}$ be the quotient functor. Consider the composite functor $T' - \otimes_A B : \mathscr{M} \to \mathscr{N}/\mathscr{N}_{r+1}$. This functor is right exact. If we can show that it preserves kernels, and so is exact, then, in view of (i), it will factor uniquely through $\mathscr{M}/\mathscr{M}_{r+1}$. So let u be a monomorphism in $\mathscr{M}$. Let $\mathfrak{Q}$ be a prime ideal of B with $ht\mathfrak{Q} \leqslant r$ and let $\mathfrak{p} = \mathfrak{Q} \cap A$. We want to show that $\text{Ker}(u \otimes B)_{\mathfrak{Q}} = 0$. But $\text{Ker}(u \otimes B)_{\mathfrak{Q}} = \text{Ker}(u_{\mathfrak{p}} \otimes B_{\mathfrak{Q}})$ and since $B_{\mathfrak{Q}}$ is $A_{\mathfrak{p}}$-flat, this is zero.

PROPOSITION 7.11. *The functor induced in Proposition* 7.10 *preserves noetherian objects.*

Proof. Let TM be a noetherian object of $\mathscr{M}/\mathscr{M}_{r+1}$. There is an epimorphism $(TA)^z \to TM$ for some integer z, so we can find an M' in M, an epimorphism $A^z \to M'$ and a homomorphism $f : M \to M'$ with $\text{Ker} f$ and $\text{Coker} f$ in $\mathscr{M}_{r+1}$. From $A^z \to M' \to 0$ exact we get $B^z \to M' \otimes_A B \to 0$ exact. The formal properties of $\otimes_A$ give $\text{Coker}(f \otimes B) = (\text{Coker} f) \otimes_A B$. Also there is a natural epimorphsim $(\text{Ker} f) \otimes_A B \to \text{Ker}(f \otimes B)$. Since $\text{Ker} f$ and $\text{Coker} f$ are in $\mathscr{M}_{r+1}$ we conclude that $\text{Ker}(f \otimes B)$ and $\text{Coker}(f \otimes B)$ are in $\mathscr{N}_{r+1}$ by (i) of the previous proposition. Thus the map $(TA)^z \otimes_A B \to TM \otimes_A B$ is an epimorphism. But $(TA)^z \otimes_A B = (T'B)^z$, so $TM \otimes_A B$ is noetherian.

PROPOSITION 7.12. *Let A be n-noetherian, and B an A-algebra which is n-noetherian and n-flat. For each integer i, $1 \leqslant i \leqslant n$, $- \otimes_A B$ induces a homomorphsim*

$$t_B : W_i(A) \to W_i(B).$$

Proof. By Propositions 7.10 and 7.11 $- \otimes_A B$ induces exact functors $(\mathcal{M}_i/\mathcal{M}_{i+1})^{\#} \to (\mathcal{N}_i/\mathcal{N}_{i+1})^{\#}$ and $(\mathcal{M}_{i+1}/\mathcal{M}_{i+1})^{\#} \to (\mathcal{N}_{i+1}/\mathcal{N}_{i+1})^{\#}$, which in turn induce homomorphisms

$$\begin{array}{ccc} K^0(\mathcal{M}_i/\mathcal{M}_{i+1}) & \to & K^0(\mathcal{M}_{i-1}/\mathcal{M}_{i+1}) \\ \downarrow & & \downarrow \\ K^0(\mathcal{N}_i/\mathcal{N}_{i+1}) & \to & K^0(\mathcal{N}_{i+1}/\mathcal{N}_{i+1}) \end{array}$$

making the diagram commutative. Then t_B is obtained by diagram chasing.

We note that if B is an A-algebra which is a flat A-module, then B satisfies condition (F_m) for all m. Hence when $W_i(A)$ and $W_i(B)$ are defined there is a homomorphism $t_B : W_i(A) \to W_i(B)$. Two cases of this phenomenon are of special importance.

If A is n-noetherian and S is a multiplicatively closed subset of A, then $B = A_S$ is a flat A-module. By Theorem 3.1, B is noetherian for the set of extensions of prime ideals of B of height at most n, and so B is n-noetherian. By the same considerations as in [*2*] we obtain

THEOREM 7.13. *Let A be n-noetherian and S a multiplicatively closed subset of A. Then A_S is n-noetherian. For each integer i, $1 \leqslant i \leqslant n$ the functor $- \otimes_A A_S$ defies an epimorphism $W_i(A) \to W_i(A_S)$. The kernel is generated by the classes of the modules $A/\mathfrak{p}$, $ht\mathfrak{p} = i$, $\mathfrak{p} \cap S \neq \emptyset$.*

The corollaries in [*2*] follow easily and we omit even their statements.

Suppose that A has the following property:

> (C_i) If $\mathfrak{P}$ is a prime ideal of A of height $k(\leqslant i)$, and if $\mathfrak{Q}/\mathfrak{P}$ is a prime ideal of $A/\mathfrak{P}$ of height at most $i - k$, then $ht\mathfrak{Q} \leqslant i$.

Suppose that A is n-noetherian and satisfies property (C_n). Let x be an element of A, which is contained in no minimal prime ideal of A such that $F(Ax) \cap A = Ax$, where F is the composite of the functors T and its adjoint S associated with the set $\mathcal{P}$ of prime ideals of A of height at most n. This condition is to say $Ax = \bigcap_{\mathfrak{p}\in\mathcal{P}} (Ax)_{\mathfrak{p}} \cap A$. If each prime ideal in $\mathrm{Ass}_{\mathcal{P}}(A/Ax)$ has height one, then by Theorem 3.3 and the condition (C_n), A/Ax is $(n-1)$-noetherian. It is clear that (under the above premises) the forgetful functor from the category $\mathcal{N}$ of A/Ax-modules to the category $\mathcal{M}$ of A-modules defines functors $\mathcal{N}_i/\mathcal{N}_{i+1} \to \mathcal{M}_{i+1}/\mathcal{M}_{i+2}$ and $\mathcal{N}_{i+1}/\mathcal{N}_{i+1} \to \mathcal{M}_i/\mathcal{M}_{i+2}$ both of which preserve the noetherian subcategories. We thus have defined a homomorphism $W_i(A/Ax) \to W_{i+1}(A)$. The statement of Theorem 7.13 yields the following.

PROPOSITION 7.14. *If $i > 0$, then the sequence*

$$W_i(A/Ax) \to W_{i+1}(A) \to W_{i+1}(A[x^{-1}]) \to 0$$

is exact.

Remark. This may be extended to $i = 0$ if $W_0(A)$ is defined to be the $C_0(A)$ of [2] rather than 0 as was done in [2].

The next case to be studied is a polynomial extension of the ring of indeterminants. Let $B = A[\{X_a\}_{a \in E}]$. By Theorem 6.4, B is n-noetherian and each prime ideal of height at most n in B is the extension of a prime ideal of the same height in a finite polynomial extension of A. For each finite subset F of E, let $B_F = A[\{X_b\}_{b \in F}]$. Then $B = \bigcup_F B_F$ and the above remark gives also $D_i(B) = \bigcup_F \mathrm{Im}(D_i(B_F) \to D_i(B))$ and hence $W_i(B) = \bigcup_F \mathrm{Im}(W_i(B_F) \to W_i(B))$. Thus if we show that $W_i(A) \to W_i(B_F)$ is an epimorphism for each such F, we can conclude that $W_i(A) \to W_i(B)$ is an epimorphism for each i, $1 \leqslant i \leqslant n$.

THEOREM 7.15. *Let A be an n-noetherian ring and $\{X_a\}$ a set of indeterminants. Let $B = A[\{X_a\}]$. For each i, $1 \leqslant i \leqslant n$ the homomorphism $W_i(A) \to W_i(B)$ induced by $- \otimes_A B$ is an epimorphism.*

Proof. In the discussion preceding the statement of the theorem we reduced the problem to showing that $W_i(A) \to W_i(B)$ is an epimorphism when B is a finite polynomial extension of A. By induction we are then reduced to showing the homomorphism is an epimorphism when $B = A[X]$, and this we proceed to do. The proof is essentially the one found in [2].

Let $\mathfrak{P}$ be a prime ideal of B with $ht\mathfrak{P} = i$. Let $\mathfrak{p} = \mathfrak{P} \cap A$. If $ht\mathfrak{p} = i$, then $\mathfrak{P} = \mathfrak{p}B$ and so the class of $B/\mathfrak{P}$ is the image of the class of $A/\mathfrak{p}$. Suppose then that $ht\mathfrak{p} = i - 1$ (the only other possibility). Let $f \in B$ be such that $\mathfrak{P}B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p} + fB_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}[X] + fA_\mathfrak{p}[X]$. Consider the ideal $I = fB + \mathfrak{p}B$. Let $\mathfrak{Q}_j$ be the $\mathcal{P}_i$ primary components of I in B with $\mathrm{Ass}_{\mathcal{P}_i}(B/\mathfrak{Q}_j) = \{\mathfrak{P}_j\}$ where $\mathfrak{P}_1 = \mathfrak{P}$. As in [2], $ht\mathfrak{P}_j = i$ and for $j \geqslant 2$, $ht(\mathfrak{P}_j \cap A) = i$. In light of Proposition 4.24 we then have $0 = [B/(fB + \mathfrak{p}B)] = \sum_{j=1}^{r} [B/\mathfrak{Q}_j] = [B/\mathfrak{P}] + \sum_{j=2}^{r} l_{\mathfrak{P}_j}((B/\mathfrak{Q}_j)_{\mathfrak{P}_j})[B/\mathfrak{P}_j]$. Hence $[B/\mathfrak{P}]$ is an image since each $\mathfrak{P}_j$ is an extended ideal.

The corollaries in [2] now follow.

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